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by

Philip G. Hodge, Jr.

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THE THEORY OF ROTATIONALLY SYMMETRIC PLASTIC SHELLS

By Philip G. Hodge, Jr.

ABSTRACT

The defining equations for a rigid/perfectly-plastic shell are derived from basic principles. On the basis of a single geometric assumption for the velocity field, generalized strain rates and stresses are defined and equilibrium relations deduced. Shell yield conditions and the flow law are discussed in general terms and then specifically for piecewise linear yield conditions.

Preceding the general shell problem, the theory of beams under bending and axial forces is discussed to give a general insight into plastic structural behavior. The paper closes with an application to cylindrical shells and a discussion of areas for future development.

1. INTRODUCTION

For historical reasons, the structural theory of shells has, until recently, been exclusively an elastic theory. Thus, in the usual development, the assumptions of a linear theory of elasticity and of shell theory have been introduced at the outset, and equations have been derived which apparently depend upon both of these sets of assumptions. As a result, it is desirable to begin an inelastic theory of shells by a consideration of basic principles. In the interests of possible other inelastic theories, the basic kinematic and static equations of shell theory have been derived in Sec. 3 by a method which is independent of any particular constitutive equations of the material. The final results will, of course, come as no surprise to those familar with classical elastic shell theory, but it is not obvious that the full extent of their generality has always been appreciated previously.

Although aiming for maximum generality of material assumptions, it has seemed desirable to limit the treatment here to a rotationally symmetri shell under rotationally symmetric loading with no torque. This limitation considerably simplifies the mathematical presentation of the fundamental concepts presented, although the concepts themselves are evidently applicable to the more general case of arbitrary shells under arbitrary loadings. Further justification lies in the fact that very few problems have been solved to date which do not fall into the rotationally symmetric catagory.* For

* The only two exceptions known to the author are a paper by Fialkow [1] in which bounds are found on a cylindrical roof, and one by Hodge and Panarelli [2] concerned with a cylindrical shell under pressure, end load, and torque. (Numbers in brackets refer to the list of references collected at the end of the paper).

brevity of notation, we shall use the unmodified term "shell" to refer to the fully rotationally symmetric catagory defined above.

By way of introduction to the shell problem, we present in Sec. 2 a development of the general and plastic theory of beams under combined bending and axial force which was first formulated by Onat and Prager [3, 4]. In this much simpler case, we can show in some detail how a structural theory is derived from a three-dimensional theory. A single kinematical assumption is made concerning the deformation or velocity field. Using this assumption and the principle of virtual work, one is led naturally to the definition of generalized strain rates and generalized stresses. As first pointed out by Prager [5], the theorems of limit analysis apply immediately to such variables; Hodge [6, 7] has shown that the general theory of elasticity and various models of plasticity can all be conveniently developed in terms of generalized variables.

When the constitutive equations of a rigid/perfectly-plastic material are inserted into the beam problem, the concept and principal properties of the yield curve and plastic flow law are all derived in a simple manner. That these properties are all valid for any number of generalized strain-rate and stress variables is certainly plausible. Although they can all be proved based upon Drucker's "stability postulate" [8, 9], it has seemed more in keeping with the spirit of the present paper to assume them and develop from them the specific plastic constitutive equations for the shell problem, as is done in Sec. 4.

The derivation of yield conditions for plastic shells has been the subject of numerous papers.* Although, the resulting yield conditions differ greatly in mathematical complexity, the theorems of

* An account of most of these, together with original source references, can be found in [10].

limit analysis can be invoked to bound their differences in predictions. Since very little direct experimental evidence is available upon which to base a reasonable choice, it has seemed advisable to concentrate attention upon the mathematically simple class of piecewise-linear yield conditions in Sec. 5. Not only does this greatly simplify the solution of problems, but, as is shown, the results can be used to bound the yield-point loads of any conceivable material yield condition.

The primary purpose of the present paper is to survey the theory of plastic shells, rather than to give a catalogue of problem solutions. However, in the interest of providing a typical application which is mathematically simple, we have included an example concerned with a cylindrical shell in Sec. 6.

The treatment in Sections 4 through 6 was intentionally restricted to an idealized rigid/perfectly-plastic material. The final section of the paper discusses the reasons for this limitation and indicates qualitatively some possible generalizations.

2. BEAM THEORY

As a prototype of shell theory, we shall first examine the much simpler case of an initially straight beam under the influence of bending moment M and direct stress N. We shall make the initial assumption that plane normal sections remain inextensible, plane, and normal to the centroidal aris. At any cross section x, then the velocity field is

$$u_x = V - zW' \quad u_y = 0 \quad u_z = W \tag{2.1}$$

where V and W are functions of x.

The strain-rate tensor field associated with (2. 1) has the axial strain as its only non-vanishing component:

$$\epsilon_x = V' - zW'' \tag{2.2}$$

so that the total internal rate of work reduces to

$$W_{int} = \int_V \sigma_{ij} \epsilon_{ij} \, dV = \int_0^L \int_A \sigma_x (V' - zW'') \, dA \, dx \tag{2.3}$$

Since V and W depend only on x, we may rewrite (2. 3) in the form

$$W_{int} = \int_0^L (NV' - MW'') \, dx \tag{2.4}$$

where

$$N = \int_A \sigma_x dA \qquad M = \int_A \sigma_x z \, dA \tag{2.5}$$

Since N and M are the stress resultants for direct stress and bending moments, respectively, they are an obvious choice for generalized stresses for the beam problem. It then follows from Prager

[5] that the generalized strain rate should be

$$e = V' \qquad K = -W'' \tag{2.6}$$

whence we can rewrite (2.2) in the form

$$\epsilon_x = e + zK \tag{2.7}$$

An alternative form of (2.4) is obtained by integrating it twice by parts:

$$W_{int} = -\int_0^L (N'V + M''W)\,dx + [NV + M'W - MW']_0^L \tag{2.8}$$

The external rate of work may be written

$$W_{ext} = \int_0^L PW dx + [\bar{N}V + \bar{S}W - \bar{M}W']_0^L \tag{2.9}$$

where P is the applied normal load and $\bar{N}$, $\bar{S}$, and $\bar{M}$ are, respectively, the axial force, shear force, and bending moments applied at the beam ends.

The principle of virtual work rate states that the internal and external work rates must be equal for all sufficiently continuous velocity fields V, W. Obviously this condition requires

$$\begin{aligned} &N' = 0 \qquad M'' + P = 0 \qquad 0<x<L \\ &N = \bar{N} \qquad M' = \bar{S} \qquad M = \bar{M} \qquad x = 0,\ x = L \end{aligned} \tag{2.10}$$

The development thus far is independent of any material property. Since only the axial force σ_x enters in the definition of the generalized stresses, we shall assume that it is the only stress to influence the material behavior. In particular, for a rigid/perfectly-plastic material, the

the yield behavior is fully characterized by the yield stress σ_0. If $\sigma_x = \sigma_0$ at an element, the element may have any positive axial strain rate ϵ_x; if $\sigma_x = -\sigma_0$ it may have any negative axial strain rate rate; if $-\sigma_0 < \sigma_x < \sigma_0$ the element must remain rigid; stress states $|\sigma_x| > \sigma_0$ are not tolerated. In other words the strain rate can be non-zero only if $\sigma = \sigma_0$ for positive ϵ_x and $\sigma = -\sigma_0$ for negative ϵ_x.

Since (2.7) describes a strain rate which is linear in z, there will be a particular value $z = \zeta H$ at which $\epsilon_x = 0$. For all z on one side ζH, ϵ_x will be positive and hence $\sigma_x = \sigma_0$, whereas on the other side, ϵ_x will be negative and hence $\sigma_x = -\sigma_0$. Assuming for definiteness that K is positive, we see that the strain-rate and stress distributions must have one of the forms shown in Fig. 1. Therefore, it follows from (2.5), (2.6), and Fig. 1 that

$$N = \sigma_0 \int_{-H}^{H} B(z)\, dz = N_0 \qquad M = 0$$

$$e/H \geq K \geq 0 \qquad \text{for} \qquad \zeta \leq -1 \tag{2.11a}$$

$$N = \sigma_0 \left[-\int_{-H}^{\zeta H} B(z)\, dz + \int_{\zeta H}^{H} B(z)\, dz \right]$$

$$M = \sigma_0 \left[-\int_{-H}^{\zeta H} z\, B(z) dz + \int_{\zeta H}^{H} z\, B(z)\, dz \right]$$

$$e/(-\zeta H) = K \geq 0 \qquad \text{for} \quad -1 \leq \zeta \leq 1 \tag{2.11b}$$

$$N = -\sigma_0 \int_{-H}^{H} B(z)\, dz = -N_0 \qquad M = 0$$

$$-e/H \geq K \geq 0 \qquad \text{for} \quad 1 \leq \zeta \tag{2.11c}$$

where B(z) is the width of the section.

Equation (2.11b) defines a curve in the N, M plane. If we consider the case $K < 0$, $-1 \leq \zeta \leq 1$, we obtain the image in the origin of (2.11b). Evidently the sum of these two curves is closed. This resulting closed curve is called the yield curve. If the stress point N, M is on the yield curve, plastic flow can take place; if the stress point is inside the curve, the section is rigid; stress points outside of the yield curve are not tolorated. Figure 2 shows the resulting yield curve for a rectangular section.

It follows from (2.11b) that

$$dM/dN = \zeta H \qquad (2.12)$$

whence the yield curve is evidently convex. Further, since

$$K/e = -1/\zeta H \qquad (2.13)$$

the "strain-rate vector" with components (e, K) is normal to the yield curve at the stress point ζ.

Equations (2.11a) give the single stress point $(N_0, 0)$ but permit a variety of strain-rate vectors. If ζ tends to -1 with positive K, then it follows from (2.13) that

$$\lim_{\zeta \to -1} (e, K) = \lambda_1 (1, 1/H) \qquad (2.14a)$$

where λ_1 is an arbitrary positive scalar. Similarly, if K is negative,

$$\lim_{\zeta \to -1} (e, K) = \lambda_2 (1, -1/H) \qquad (2.14b)$$

where $\lambda_2 \geq 0$. From the inequality (2.11a) on e/H and K, together with the corresponding inequality

$$e/H \geq -K \geq 0$$

for negative K, we see that at $\zeta = -1$, e and K are restricted only by

$$e/H \geq |K| \geq 0 \tag{2.15}$$

Finally, it is obvious that the sum of the two vectors (2.14) will satisfy (2.15) for any non-negative choice of λ_1 and λ_2. Typical strain-rate vectors are shown in Fig. 2 for a regular point B governed by (2.11b) and for a singular point A governed by (2.11a).

For the beam problem considered here, we have started with a simple geometrical assumption and with the tensile behavior of the material. We have then developed the equilibrium conditions, the convexity of the yield curve, the normality of the strain-rate vector of a regular point, and the behavior of the strain-rate vector at a singular point. It can be shown that these concepts are all carried over to the more general problem of rotationally symmetric plastic shells. We shall use these facts in the next two sections.

3. BASIC EQUATIONS OF SHELL THEORY

The theory of elastic shells has been thoroughly studied, whereas investigations of inelastic shells are quite recent. As a result, it is not always clear which aspects of elastic shell theory are dependent upon the elastic constitutive equations and which are equally valid for other materials. Therefore, we shall begin this discussion by deriving the fundamental rotationally symmetric shell equations in a manner which makes no reference to any constitutive equations. The results will thus be applicable to shells of any material and hence, in particular, to a rigid/perfectly-plastic material.

A single kinematic assumption is made concerning the type of deformation which the shell undergoes, namely:

> Initially straight normals to
> the undeformed middle surface
> remain straight, inextensible, (3.1)
> and normal to the deformed
> middle surface.

The degree of validity of this assumption as applied to a real physical structure determines the validity of calling the structure a shell in the sense used in this paper. The validity of (3.1) cannot be checked internally on the basis of shell theory but must be done on the basis of complete three-dimensional solutions, experimental evidence, and/or sound engineering judgement. It is certainly conceivable that its validity should depend not only on the geometrical parameters such as thickness and radii of curvature, but on the type of material being considered, i.e., elastic, plastic, etc.

A detailed discussion of the situations under which (3. 1) is a reasonable approximation to reality would be beyond the scope of the present paper, even if it could be done in any generality. Therefore, in the remainder of the present development we shall assume (3. 1) and investigate its consequences.

The method of attack is similar to that used in the previous section. We shall first find the expressions for linear strain-rate components for a general deformation of a shell. Next we particularize these strain-rate velocity relations in accord with assumption (3. 1). The internal work rate associated with an arbitrary displacement field subject to (3.1) is then computed and shown to naturally suggest the generalized strain-rate variables appropriate to the rotationally symmetric shell problem. Finally, the principle of virtual work is invoked to define the generalized stress variables and to obtain the equilibrium equations which they must satisfy.

We consider an arbitrary longitudinal plane of the shell and let O be an arbitrary point on the center line, Fig. 3. The coordinate ϕ is defined as the angle between the axis and a normal to the center line through O; the coordinate ξ is measured inward along the normal from the center line. Let points P and R have the same ϕ coordinate as O with ξ coordinates ξ and $\xi + \Delta\xi$, respectively; let Q have coordinates $\phi + \Delta\phi$ and ξ, all as shown in Fig. 3.

It is now convenient to introduce a set of Cartesian coordinates r, z in the plane of Fig. 3. The following vectors may be readily identified:

$$\overrightarrow{PQ} = (R_1 - \xi)\Delta\phi(\cos\phi, \sin\phi)$$
$$\overrightarrow{PR} = \Delta\xi \ (-\sin\phi, \cos\phi) \tag{3.2}$$

Consider next a small vector velocity field $\underset{\sim}{u}$. If u_n and u_ϕ are the components of $\underset{\sim}{u}$ in the ξ and ϕ directions, then $\underset{\sim}{u}$ may be referred to r and z components by

$$\underset{\sim}{u} = u_n \ (-\sin\phi, \cos\phi) + u_\phi \ (\cos\phi, \sin\phi) \tag{3.3}$$

Let P', Q', and R' denote the respective positions of P, Q, and R after the displacement. Then

$$\overrightarrow{P'Q'} = \overrightarrow{PQ} + \overrightarrow{QQ'} - \overrightarrow{PP'} = \overrightarrow{PQ} + \underset{\sim}{u}(Q) - \underset{\sim}{u}(P)$$

hence, in view of Eqs. (3.2) and (3.3)

$$\overrightarrow{P'Q'} = (R_1 - \xi)\ \Delta\ \phi[(1 + \frac{u_\phi' - u_n}{R_1 - \xi})\ (\cos\phi, \sin\phi) + (u_\phi + u_n')\ (-\sin\phi, \cos\phi)]$$
$$\overrightarrow{P'R'} = \Delta\xi\ [(1 + \dot{u}_n)\ (-\sin\phi, \cos\phi) + \dot{u}_\phi\ (\cos\phi, \sin\phi)] \tag{3.4}$$

Here we have regarded $\Delta\phi$ and $\Delta\xi$ as infinitesimals and have denoted differentiation with respect to ϕ and ξ by a prime and a dot, respectively.

If $|\underset{\sim}{u}/R_1|$, $|\underset{\sim}{u}'/R_1|$, and $|\dot{\underset{\sim}{u}}|$ are all small compared to unity we can easily deduce from (3.4) and (3.2) that

$$|P'Q'| = (R_1 - \xi)\Delta\phi(1 + \frac{u_\phi' - u_n}{R_1 - \xi})$$
$$|P'R'| = \Delta\xi\ (1 + \dot{u}_n) \tag{3.5}$$
$$\overrightarrow{P'Q'} \cdot \overrightarrow{P'R'} = \Delta\phi\Delta\xi\ (R_1 - \xi)\ (\dot{u}_\phi + \frac{u_\phi + u_n'}{R_1 - \xi})$$

$$|PQ| = (R_1 - \xi)\Delta\phi \qquad |PR| = \Delta\xi \qquad \overrightarrow{PQ} \cdot \overrightarrow{PR} = 0$$

Extensional strain rate is defined as the rate of change in length per unit length of a line element. Therefore the extensional strain rate of an element in the direction of $\overrightarrow{PR}$ is

$$\epsilon_n = \frac{|P'R'| - |PR|}{|PR|} = \dot{u}_n \qquad (3.6a)$$

and, in similar fashion

$$\epsilon_\phi = \frac{|P'Q'| - |PQ|}{|PQ|} = \frac{u_\phi{}' - u_n}{R_1 - \xi} \qquad (3.6b)$$

The extensional strain rate in the circumferential direction is the rate of increase in the circumference of a circle through P divided by the original circumference. Since the circumference is proportional to the radius, we may equally well use the r components of the vectors $\overrightarrow{CP}$ and $\overrightarrow{CP'}$ to define

$$\epsilon_\theta = \frac{(\overrightarrow{CP'})_r - (\overrightarrow{CP})_r}{(\overrightarrow{CP})_r} = \frac{u_\phi \cos_\phi - u_n \sin\phi}{(R_2 - \xi)\sin\phi} \qquad (3.6c)$$

The shear strain rate is defined as the rate of change in an initially right angle. In view of the restriction to rotational symmetry

$$\gamma_{n\theta} = \gamma_{\phi\theta} = 0 \qquad (3.6d)$$

whereas the remaining shear strain rate is

$$\gamma_{n\phi} = \frac{\overrightarrow{P'Q'} \cdot \overrightarrow{P'R'}}{|P'Q'||P'R'|} = \dot{u}_\phi + \frac{u_\phi + u_n{}'}{R_1 - \xi} \qquad (3.6e)$$

Equations (3.6) give the strain rate components at any point in the shell for an arbitrary displacement field. We now introduce the assumption (3.1). If V_n and V_ϕ represent the velocities of a point O on the center line, then it follows from Fig. 4 that at P

$$u_n = V_n \qquad u_\phi = V_\phi - \frac{\xi}{R_1}(V_\phi + V_n') \tag{3.7}$$

Substituting (3.7) into (3.6) we find that for the particular velocity field being considered,

$$\epsilon_n = \gamma_{n\theta} = \gamma_{\phi\theta} = \gamma_{n\phi} = 0$$

$$\epsilon_\theta = \frac{V_\phi \cot\phi - V_n}{R_2 - \xi} - \frac{\xi \cot\phi}{R_2 - \xi} \cdot \frac{V_\phi + V_n'}{R_1} \tag{3.8}$$

$$\epsilon_\phi = \frac{V_\phi' - V_n}{R_1 - \xi} - \frac{\xi}{R_1 - \xi} \cdot \left(\frac{V_\phi + V_n'}{R_1}\right)'$$

The internal rate of work associated with a rotationally symmetric shell may be written

$$W_{int} = \int_\beta^\alpha \int_{-H}^{H} \int_0^{2\pi} w_{int}(R_1 - \xi) \sin\phi \, d\theta \, d\xi \, d\phi \tag{3.9}$$

where the shell extends from $\phi = \beta$ to $\phi = \alpha$ and is of thickness 2H; here

$$w_{int} = \sigma_n \epsilon_n + \sigma_\theta \epsilon_\theta + \sigma_\phi \epsilon_\phi + \tau_{n\phi}\gamma_{n\phi} + \tau_{\theta\phi}\gamma_{\theta\phi} + \tau_{\theta n}\gamma_{\theta n}$$

We substitute the particular strain rate field (3.8) into (3.9) and note that V_n, V_ϕ, R_1, and R_2 are independent of ξ and θ to write the resulting expression in the form

$$W_{int} = 2\pi \int_{\beta}^{\alpha} \Bigg\{ \left[\frac{V_\phi \cot \phi - V_n}{R_2} \right] \cdot \left[\frac{1}{2\pi} \int_{-H}^{H} \int_{0}^{2\pi} \sigma_\theta \frac{R_1 - \xi}{R_1} \, d\theta \, d\xi \right]$$

$$+ \left[\frac{V_\phi' - V_n}{R_1} \right] \cdot \left[\frac{1}{2\pi} \int_{-H}^{H} \int_{0}^{2\pi} \sigma_\phi \frac{R_2 - \xi}{R_2} \, d\theta \, d\xi \right]$$

$$+ \left[- \frac{\cot \phi}{R_2} \left(\frac{V_\phi + V_n'}{R_1} \right) \right] \cdot \left[\frac{1}{2\pi} \int_{-H}^{H} \int_{0}^{2\pi} \xi \sigma_\theta \frac{R_1 - \xi}{R_1} \, d\theta \, d\xi \right]$$

$$+ \left[- \frac{1}{R_1} \left(\frac{V_\phi + V_n'}{R_1} \right)' \right] \cdot \left[\frac{1}{2\pi} \int_{-H}^{H} \int_{0}^{2\pi} \xi \sigma_\phi \frac{R_2 - \xi}{R_2} \, d\theta \, d\xi \right] \Bigg\} R_1 R_2 \sin \phi \, d\phi \tag{3.10}$$

The factors in the first bracket of each term of (3.10) are chosen as our generalized strain rates. They may be given simple physical interpretations as extension rates and approximate curvature rates of the middle surface. Thus

$$e_\theta = (V_\phi \cot \phi - V_n)/R_2 \qquad e_\phi = (V_\phi' - V_n)/R_1$$

$$K_\theta = - \frac{\cot \phi}{R_2} \left(\frac{V_\phi + V_n'}{R_1} \right) \qquad K_\phi = - \frac{1}{R_1} \left(\frac{V_\phi + V_n'}{R_1} \right)' \tag{3.11a}$$

As first pointed out by Prager [5], generalized stresses and strain rates must be chosen so that the internal work rate is proportional to their scalar product. Therefore, we define the second bracketed factor of each term as a generalized stress:

$$N_\theta = \int_{-H}^{H} \sigma_\theta \frac{R_1 - \xi}{R_1} d\xi \qquad N_\phi = \int_{-H}^{H} \sigma_\phi \frac{R_2 - \xi}{R_2} d\xi$$

$$M_\theta = \int_{-H}^{H} \xi\sigma_\theta \frac{R_1 - \xi}{R_1} d\xi \qquad M_\phi = \int_{-H}^{H} \xi\sigma_\phi \frac{R_2 - \xi}{R_2} d\xi \tag{3.11b}$$

where we have assumed for simplicity that the stress distribution also is rotationally symmetric.

We observe that (3.11) are precisely the total force per unit length and total moment per unit length about the center line which are transmitted across an element of the shell. Therefore, the definitions (3.11) are in agreement with the usual definitions of stress resultants.

External work may be done on the shell by surface loads applied at $\xi = \pm H$, by edge loads applied at $\phi = \beta$ and $\phi = \alpha$, or by body forces. We denote by T_n and T_ϕ the force per unit area components transmitted across a surface ξ = const. from greater to lesser values of ξ, by S_n and S_ϕ similar forces across a surface ϕ = const., and by F_n and F_ϕ the body force per unit volume components. Then the external rate at which work is done during an arbitrary symmetric deformation is

$$\begin{aligned} W_{ext} = \int_0^{2\pi} \Big\{ & \Big[\int_{-H}^{H} (S_n u_n + S_\phi u_\phi)(R_2 - \xi) \sin\phi \, d\xi \Big]_\beta^\alpha \\ & + \Big[\int_\beta^\alpha (T_n u_n + T_\phi u_\phi)(R_1 - \xi)(R_2 - \xi) \sin\phi \, d\phi \Big]_{-H}^{H} \\ & + \int_\beta^\alpha \int_{-H}^{H} (F_n u_n + F_\phi u_\phi)(R_1 - \xi)(R_2 - \xi) \sin\phi \, d\xi \, d\phi \Big\} d\theta \end{aligned} \tag{3.12}$$

We substitute the particular velocity field (3. 7) into (3. 12) and rearrange the terms to obtain

$$\frac{W_{ext}}{2\pi} = \Bigg[\Bigg\{ V_n \left(\int_{-H}^{H} S_n \frac{R_2 - \xi}{R_2} d\xi \right) + V_\phi \left(\int_{-H}^{H} S_\phi \frac{R_2 - \xi}{R_2} d\xi \right)$$

$$- \frac{V_\phi + V_n'}{R_1} \left(\int_{-H}^{H} \xi S_\phi \frac{R_2 - \xi}{R_2} d\xi \right) \Bigg\} \quad R_2 \sin\phi \Bigg]_\beta^\alpha$$

$$+ \int_\beta^\alpha \Bigg\{ V_n \left(\left[T_n \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} \right]_{-H}^{H} + \int_{-H}^{H} F_n \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} d\xi \right)$$

$$+ V_\phi \left(\left[T_\phi \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} \right]_{-H}^{H} + \int_{-H}^{H} F_\phi \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} d\xi \right)$$

$$- \frac{V_\phi + V_n'}{R_1} \left(\left[\xi T_\phi \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} \right]_{-H}^{H} + \int_{-H}^{H} \xi F_\phi \frac{R_1 - \xi}{R_1} \frac{R_2 - \xi}{R_2} d\xi \right) \Bigg\} R_1 R_2 \sin\phi \, d\phi \qquad (3.13)$$

Just as the factors in (3. 10) were recognized as resultant internal forces and moments, so the factors in parentheses in (3. 13) may be recognized as resultant applied forces and couples. Thus we are led to define boundary forces

$$\overline{S} = \int_{-H}^{H} S_n \frac{R_2 - \xi}{R_2} d\xi \qquad \overline{N}_\phi = \int_{-H}^{H} S_\phi \frac{R_2 - \xi}{R_2} d\xi$$

$$\overline{M}_\phi = \int_{-H}^{H} \xi S_\phi \frac{R_2 - \xi}{R_2} d\xi \qquad (3.14)$$

at the edges $\phi = \beta$ and $\phi = \alpha$ of the shell, and distributed loads and couple by

$$P_n = \left[T_n \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2}\right]_{-H}^{H} + \int_{-H}^{H} F_n \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2} d\xi$$

$$P_\phi = \left[T_\phi \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2}\right]_{-H}^{H} + \int_{-H}^{H} F_\phi \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2} d\xi$$

$$C_\phi = \left[\xi T_\phi \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2}\right]_{-H}^{H} + \int_{-H}^{H} \xi F_\phi \frac{R_1 - \xi}{R_1} \quad \frac{R_2 - \xi}{R_2} d\xi$$

(3.14b)

for $\beta < \phi < \alpha$. The definitions (3.14) are consistent with the usual physically based definitions and include the possibility of a couple C_ϕ being distributed over the surface.

We next set the internal rate of work (3.10) equal to the external rate of work (3.13), using the definitions (3.11) and (3.14). Integrating the term involving M_ϕ by parts we can write the resulting equation in the form

$$\frac{1}{2\pi} (W_{ext} - W_{int}) = \int_\beta^\alpha \Big\{ V_\phi (-R_1 N_\theta \cos\phi + R_1 R_2 P_\phi \sin\phi) - V_\phi' R_2 N_\phi \sin\phi$$

$$+ V_n (R_1 N_\theta \sin\phi + R_2 N_\phi \sin\phi + R_1 R_2 P_n \sin\phi)$$

$$- \frac{V_\phi + V_n'}{R_1} \left[- R_1 M_\theta \cos\phi + (R_2 M_\phi \sin\phi)' + R_1 R_2 C_\phi \sin\phi\right] \Big\} d\phi$$

$$+ \left[\left\{V_n \bar{S} + V_\phi \bar{N}_\phi + \frac{V_\phi + V_n'}{R_1} (M_\phi - \bar{M}_\phi)\right\} R_2 \sin\phi\right]_\beta^\alpha = 0$$

(3.15)

The group of terms in the bracket in (3. 15) suggest the definition of a quantity S by

$$R_1 R_2 S \sin\phi = (R_2 M_\phi \sin\phi)' - R_1 M_\theta \cos\phi + R_1 R_2 C_\phi \sin\phi \tag{3.16}$$

whence a further integration by parts of (3. 15) leads to

$$\begin{aligned}\int_\beta^\alpha \{ & V_\phi [(R_2 N_\phi \sin\phi)' - R_1 N_\theta \cos\phi - R_2 S \sin\phi + R_1 R_2 P_\phi \sin\phi] \\ & + V_n [(R_2 S \sin\phi)' + R_1 N_\theta \sin\phi + R_2 N_\phi \sin\phi + R_1 R_2 P_n \sin\phi]\} \, d\phi \\ & = [\{V_n (S - \bar{S}) + V_\phi (N_\phi - \bar{N}_\phi) - \frac{V_\phi + V_n'}{R_1} (M_\phi - \overline{M}_\phi)\} R_2 \sin\phi]_\beta^\alpha\end{aligned} \tag{3.17}$$

If Eq. (3.17) is to be valid for all sufficiently continuous velocity functions V_ϕ and V_n, then the two factors in square brackets on the left-hand side must vanish indentically in ϕ and the three parenthentical factors on the right-hand side must vanish at the shell edges. Thus

$$\begin{aligned}&(R_2 N_\phi \sin\phi)' - R_1 N_\theta \cos\phi - R_2 S \sin\phi + R_1 R_2 P_\phi \sin\phi = 0 \\ &(R_2 S \sin\phi)' + R_1 N_\theta \sin\phi + R_2 N_\phi \sin\phi + R_1 R_2 P_n \sin\phi = 0\end{aligned} \tag{3.18}$$

for all ϕ, and at $\phi = \alpha$ and $\phi = \beta$

$$S = \bar{S} \qquad N_\phi = \bar{N}_\phi \qquad M_\phi = \overline{M}_\phi \tag{3.19}$$

Equations (3. 16) and (3. 18) are the familiar equations of equilibrium for a rotationally symmetric shell (in the usual case $C_\phi = 0$), and (3.19) express the fact that shear force, normal direct stress, and normal bending moment must each be continuous at the boundary.

In the following, it will prove convenient to deal exclusively with dimensionless quantities. To this end we denote by N_0 the maximum direct stress which the shell can withstand in uniaxial tension and by M_0 the maximum uniaxial bending moment and define

$$n_\theta = N_\theta / N_0 \qquad n_\phi = N_\phi / N_0$$
$$m_\theta = M_\theta / M_0 \qquad m_\phi = M_\phi / M_0 \tag{3.20a}$$
$$\kappa_\theta = (M_0/N_0)K_\theta \qquad \kappa_\phi = (M_0/N_0)K_\phi$$

Further we let A represent a typical dimension of the shell and define

$$r = R/A \qquad r_1 = R_1/A \qquad r_2 = R_2/A$$
$$v_n = V_n/A \qquad v_\phi = V_\phi/A \qquad h = M_0/AN_0$$
$$p_n = P_n A/N_0 \qquad p_\phi = P_\phi A/N_0 \qquad s = S/N_0$$
$$w_{int} = W_{int}/2\pi N_0 A^2 \qquad w_{ext} = W_{ext}/2\pi N_0 A^2 \tag{3.20b}$$

In terms of these dimensionless quantities, the generalized strain rates and stresses (3.11) become

$$e_\theta = \frac{v_\phi \cot\phi - v_n}{r_2} \qquad e_\phi = \frac{v_\phi' - v_n}{r_1} \tag{3.21a}$$
$$\kappa_\theta = -\frac{h\cot\phi}{r_2}\left(\frac{v_n' + v_\phi}{r_1}\right) \qquad \kappa_\phi = -\frac{h}{r_1}\left(\frac{v_n' + v_\phi}{r_1}\right)'$$

$$n_\theta = \frac{1}{N_0}\int_{-H}^{H} \sigma_\theta \frac{r_1 - \xi/A}{r_1}\, d\xi \qquad n_\phi = \frac{1}{N_0}\int_{-H}^{H} \sigma_\phi \frac{r_2 - \xi/A}{r_2}\, d\xi$$

$$m_\theta = \frac{1}{M_0}\int_{-H}^{H} \xi\sigma_\theta \frac{r_1 - \xi/A}{r_1}\, d\xi \qquad m_\phi = \frac{1}{M_0}\int_{-H}^{H} \xi\sigma_\phi \frac{r_2 - \xi/A}{r_2}\, d\xi$$

(3.21b)

and the equilibrium equations (3. 16) and (3. 18) (with $C_\phi = 0$) may be written

$$
\begin{aligned}
(rn_\phi)' - r_1 n_\theta \cos\phi - rs + rr_1 p_\phi &= 0 \\
(rs)' + r_1 n_\theta \sin\phi + rn_\phi + rr_1 p_n &= 0 \qquad (3.22) \\
h\,[\,(rm_\phi)' - r_1 m_\theta \cos\phi] - rr_1 s &= 0
\end{aligned}
$$

where

$$r = r_2 \sin\phi \qquad (3.23)$$

is the dimensionless distance from the axis.

4. CONSTITUTIVE EQUATIONS OF PLASTIC SHELL THEORY

The basic unknowns of shell theory are the four generalized stresses n_θ, n_ϕ, m_θ, m_ϕ, the shear stress s, and the velocity components v_ϕ and v_n. Since Eqs. (3.16) and (3.18) provide only three equations for the seven unknowns, it is necessary to provide four more equations in these same unknowns which characterize the particular shell material.

For a linear elastic material these constitutive equations are trivally derived. Hooke's law is substituted into (3.11b) to give generalized stresses in terms of material strains, and these latter are expressed in terms of displacements by expressions analogous to (3.8). It is timesaving to express the results in terms of generalized strains by utilizing the displacement form of (3.21a).

For the plastic material, it is necessary to first express the yield condition in terms of the generalized stresses. For any assumed form of the material yield condition, the shell yield condition can be found, at least in theory. Assuming that H/R_1 and H/R_2 are negligible compared to unity, Onat and Prager [11] have found the yield condition for a uniform shell whose material satisfies the Tresca yield condition. Similar treatments of the Mises yield condition applied to shell problems have been carried out by Hodge and Panarelli [12, 2]. However, the results are so complex as to have been relatively little used in the solution of problems. Among the few complete solutions (as opposed to bounds found by limit analysis) are those by Hopkins and Wang [13] for a flat plate, Hodge and Sawczuk [14] for a cylindrical shell, and Onat and Lance [15, 16] for a nearly flat cone.

In view of the difficulty of obtaining complete solutions according to the Tresca or Mises uniform shell yield conditions, considerable interest has been expressed in the use of approximate yield conditions. An approximate yield condition can be viewed in either of two lights. On the one hand, by means of the Bounding Surface Lemma [10] of limit analysis, it may be used to provide upper and lower bounds on the yield-point load according to some more exact yield condition. General factors for this use have been found by Hodge and Sankaranarayanan [17, 18]. Alternatively, one may visualize an ideal shell made of a modified material such that the approximate yield condition for the real shell becomes the exact yield condition for the ideal shell. If the material difference between the real and ideal shell are not too great, then the behavior of the ideal shell should provide valuable information concerning the behavior of the real shell.

Further justification for the use of approximate yield conditions for shells is derived from the fact that either the Tresca or Mises material yield condition already represents an approximation to reality, and hence it is misleading to speak of a shell yield condition based on either of them as "exact".

Once the yield condition has been decided upon, it provides one of the necessary four constitutive equations. The remaining equations are obtained from the flow law. The plastic potential flow law was first proposed by Mises [19] and later shown to be a consequence of Drucker's postulates for a stable material [8, 9]. In terms of the generalized stresses and strains defined by (3.21), we may represent the yield condition as a surface in a four-dimensional generalized stress space in the form

$$f\ (n_\theta,\ n_\phi,\ m_\theta,\ m_\phi) = 1 \tag{4.1}$$

The flow law then states that the strain rate-vector

$$\underset{\sim}{q} = (e_\theta,\ e_\phi,\ \kappa_\theta,\ \kappa_\phi) \tag{4.2}$$

must be directed along the outward normal to the yield surface at the stress point. Thus

$$\underset{\sim}{q} = \lambda\ \nabla f \tag{4.3}$$

where λ is a non-negative scalar. Since λ is unknown Eq. (4.3) is equivelent to three scalar equations in terms of the unknowns n_θ, n_ϕ, m_θ, m_ϕ, v_ϕ, v_n, as required.

At some points of the yield surface, the normal may not be uniquely defined. If the stress point is in a "crease" formed by the intersection of two smooth surfaces

$$f_1 = 1 \qquad\qquad f_2 = 1 \tag{4.4}$$

then the strain-rate vector can be any combination with non-negative coefficients of the normals to the two surfaces forming the crease:

$$\underset{\sim}{q} = \lambda_1\ \nabla f_1 + \lambda_2\ \nabla f_2 \tag{4.5}$$

Observe that in this case Eqs. (4.4) for the yield condition provide two constitutive equations; since λ_1 and λ_2 are now independent unknowns, (4.5) provides only the necessary two remaining equations.

In degenerate cases the above reasoning may be generalized to higher order intersections which result in more yield conditions of the form (4.4) and more unknown λ's in the flow law. However, we observe that Eqs. (3.22) and (4.4) provide five equations containing only n_θ, n_ϕ, m_θ, m_ϕ, and s as unknowns, and that the velocity unknowns v_n and v_ϕ occur only in the two independent equations of (4.5). Therefore, any hypothesis which added to (4.4) at the expense of (4.5) would generally lead to over-determined stresses and under-determined velocities and hence be unsolvable.

The above reasoning has an interesting result in the case of a piecewise-linear yield surface. If (4.1) represents a linear surface, then ∇f will be a constant. It follows that (3.22) and (4.1) will provide four equations for five stress variables whereas (4.3) will provide three equations containing only v_ϕ and v_n. Therefore, except possibly in certain degenerate cases, if the yield condition is piecewise linear, the constitutive equations must be of the form (4.4) and (4.5).

5. PIECEWISE LINEAR YIELD CONDITIONS

Based upon previous work by Hill [20] and Ivlev [21], Haythornthwaite [22] has shown that any material yield condition may be conveniently bounded by two piecewise linear material yield conditions when the only physical information is a single test measurement. We consider a representation in principle stress space as shown in Fig. 5 and suppose the point A to be a measured tensile yield stress σ_0. Following Haythornthwaite, we assume only that the material is isotropic with equal tensile and compressive yield stresses, that it satisfies Drucker's stability postulate, [8, 9], and that the sharply defined rigid-perfectly plastic yield-point load represents useful information. It follows that the yield surface must be convex with symmetry every 30°. Therefore, if the tensile yield stress σ_0 is known, the yield surface must pass through point A in Fig. 5 and must lie between the solid and dashed curves. The inner solid curve is Tresca's [23] condition of maximum shearing stress

$$\max [\, |\sigma_1 - \sigma_2|,\ |\sigma_2 - \sigma_3|,\ |\sigma_3 - \sigma_1| \,] = \sigma_0 \qquad (5.1)$$

whereas the outer dashed curve is Hill's [20] condition of maximum reduced stress

$$\max |\sigma_i - (\sigma_1 + \sigma_2 + \sigma_3)/3| = \sigma_0 \qquad i = 1, 2, 3 \quad (5.2)$$

In the analysis of thin plates or shells where the state of stress is approximately plane, the information in Fig. 5 may be represented in a σ_1, σ_2 plane. For flat plates under bending, the yield relation between principal bending moments is the same as that between principal stresses, so that Haythornthwaite [22] was able to use the bounding surface lemma to bound the yield-point load of a circular plate under uniform pressure.

If both bending moments and direct stresses are present, as is generally the case for curved shells, the yield surface will no longer have the simple character it does in a σ_1, σ_2 space. Rather, four stress dimensions will be required, and, due to the differences in integration formulas for moments and direct stresses, the yield surface will generally be non-linear even if the material yield condition is piecewise linear.

One method of regaining a piecewise-linear problem is to approximate the uniform shell by an idealized sandwich one. An idealized sandwich shell consists of two sheets each of thickness t', separated by a core of thickness h'. The sheets each have a tensile yield stress σ_0' and carry no shear; the core has no in-plane strength but is sufficiently strong in transverse shear to maintain separation of the sheets. The sheet thickness t' is assumed sufficiently small so that the stresses do not vary across each sheet.

If $\sigma_\beta^{\pm}$ (β= 1, 2) denote the principal stresses in the top and bottom sheet respectively, the resultant forces and moments are

$$N_\beta = (\sigma_\beta^- + \sigma_\beta^+)\, t' \qquad M_\beta = \tfrac{1}{2}(\sigma_\beta^- - \sigma_\beta^+)\, h't' \qquad (5.3)$$

Evidently Eqs. (5.3) can be solved for the four stresses $\sigma_\beta^{\pm}$ as linear functions of the forces N_β and moments M_β. It follows that any material yield condition which is piecewise linear in the stresses will be piecewise linear in N_β and M_β. Therefore, it can be represented as a convex polyhedron in a four dimensional generalized stress space with axes N_β and M_β.

On the other hand, for a uniform shell of thickness h and yield stress σ_0, the stress resultants are

$$N_\beta = \int_{-h/2}^{h/2} \sigma_\beta \, dz \qquad M_\beta = \int_{-h/2}^{h/2} \sigma_\beta \, z dz \tag{5.4}$$

At least in theory, Eqs. (5.4) can be solved for the stresses and substituted into the material yield condition to obtain the yield surface in generalized stress space. Whether the material yield condition is piecewise linear or not, the yield surface will be convex but will not generally be a polyhedron.

We wish to investigate the relation between the yield surface of the uniform shell and the yield polyhedron of the sandwich shell. To this end we first note that if a single force or single moment is acting and produces yield, then its magnitude is

$$N_0' = 2\sigma_0' t' \qquad M_0' = \sigma_0 h' t'$$

for the sandwich shell and

$$N_0 = \sigma_0 h \qquad M_0 = (1/4)\, \sigma_0 h^2$$

for the uniform shell.

The usual method of choosing the sandwich shell parameters is so that [7],

$$N_0' = N_0 \qquad M_0' = M_0 \tag{5.5}$$

i. e.

$$\sigma_0^{'} t^{'} = \tfrac{1}{2}\sigma_0 h \qquad h^{'} = \tfrac{1}{2}h \tag{5.6}$$

Now, it is evident that the yield polyhedron is fully specified by its vertices and that, regardless of the yield condition, each vertex is specified by giving values of the four pure numbers $N_\beta/N_0^{'}$, $M_\beta/M_0^{'}$. Therefore, if (5.5) holds and if the same material yield condition is used for the sandwich and uniform shells, all vertices of the yield polyhedron will lie on the yield surface. Since the polyhedron and yield surface are both convex, it follows that the polyhedron must lie inside the yield surface. Therefore, the yield polyhedron for the sandwich shell defined by (5.6) will provide a lower bound on the yield-point load for a uniform shell with the same material yield condition.

A different result is obtained if we visualize taking the uniform shell, leaving its thickness unchanged, but compressing all of its material into its top and bottom surfaces. Thus

$$h^{'} = h \qquad \sigma_0^{'} t^{'} = \tfrac{1}{2}\sigma_0 h \tag{4.7}$$

whence

$$N_0^{'} = N_0 \qquad M_0^{'} = 2M_0 \tag{4.8}$$

For any state of stress this sandwich shell will be as strong as the uniform shell in resisting moments. It follows that the resulting yield polyhedron must lie outside of the yield surface. Therefore, the sandwich shell defined by (5.7) will provide an upper bound for a uniform shell with the same material yield condition.

It is convenient to define dimensionless generalized stresses in terms of the parameters of the uniform shell by

$$n_\beta = N_\beta / N_0 = (\sigma_\beta^- + \sigma_\beta^+)/(2\sigma_0')$$
$$m_\beta = M_\beta / M_0 = (\sigma_\beta^- - \sigma_\beta^+)/(2k\sigma_0') \qquad (5.9)$$

where

$$k = 1 \text{ for the shell of } (5.6)$$
$$k = \tfrac{1}{2} \text{ for the shell of } (5.7) \qquad (5.10)$$

The stresses can then be written in the form

$$\sigma_\beta^\pm / \sigma_0' = n_\beta \pm k m_\beta \qquad (5.11)$$

Substitution of (5.11) into the yield condition will give a circumscribing polyhedron for $k = \frac{1}{2}$ and an inscribing one for $k = 1$.

An alternative circumscribing polyhedron can be obtained without reference to a sandwich shell. We observe that if the stresses in a uniform shell must satisfy a set of linear inequalities of the form $L_i(\sigma_\beta/\sigma_0) \le 0$, then the generalized stresses must satisfy

$$L_i(n_\beta) \le 0 \qquad L_i(m_\beta) \le 0 \qquad (5.12)$$

together with further inequalities which represent the interaction between force and moment. Since the yield surface is constrained by (5.12) as well as further inequalities, it follows that the convex polyhedron (5.12) must circumscribe the yield surface. We shall refer to it as the limited interaction polyhedron.

For plane stress $\sigma_3 = 0$, Tresca's yield condition (5.1) reduces to

$$\max\ [\,|\sigma_1|,\ |\sigma_2|,\ |\sigma_1 - \sigma_2|\,]/\sigma_0' = 1 \qquad (5.13)$$

This must be satisfied by both the top and bottom sheets of the sandwich shell. Therefore, it follows from (5. 11) with $k = 1$, that

$$\max [\, |n_\beta| + |m_\beta| ,\ |n_1 - n_2| + |m_1 - m_2| \,] = 1 \qquad (5.14)$$

is a lower bound for any material yield condition with tensile yield stress $\sigma_0 = \sigma_0'$.

For the maximum reduced stress condition with $\sigma_3 = 0$ we obtain

$$\max \left[\, |\sigma_\alpha - \tfrac{1}{2}\sigma_\beta| ,\ \tfrac{1}{2}|\sigma_1 + \sigma_2| \,\right] / \sigma_0' = 1 ;\ \alpha, \beta = 1, 2; \alpha \neq \beta \qquad (5.15)$$

whence, in view of (5. 11) with $k = \frac{1}{2}$

$$\max [\, |n_\alpha - \tfrac{1}{2}n_\beta| + \tfrac{1}{2} |m_\alpha - \tfrac{1}{2}m_\beta| ,\ \tfrac{1}{2}|n_1 + n_2| + \tfrac{1}{4} |m_1 + m_2| \,] = 1 \qquad (5.16)$$

The limited interaction polyhedron in this case is

$$\max [\, |n_\alpha - \tfrac{1}{2}n_\beta| ,\ |m_\alpha - \tfrac{1}{2}m_\beta| ,\ \tfrac{1}{2}|n_1 + n_2| ,\ \tfrac{1}{2}|m_1 + m_2| \,] = 1 \qquad (5.17)$$

Either of (5. 16) or (5. 17) provide upper bounds for any material yield condition.

As discussed in [22] and [24], a similar analysis in which the roles of Tresca's and Hill's material conditions are essentially interchanged may be carried out for the case in which the single experimental number is the shearing yield stress.

6. EXAMPLE: CYLINDRICAL SHELL

For an axially symmetrically loaded circular cylindrical shell we choose the axial direction x and circumferential direction θ as principal directions 1 and 2 respectively. According to the usual assumptions of thin cylindrical shell theory, $m_2 = m_\theta$ is a reaction and can be eliminated from the problem. Further, if axial load is applied only at the end of the shell, the axial stress $n_x = t$ is constant along the shell and can be regarded as a parameter. Thus, we can reduce the various polyhedra to polygons in an $n_\theta = n$, $m_x = m$ space.

We consider first the form of the Tresca polygon. The inequalities implied by (5.14) can be written

$$\left.\begin{array}{l} -1-t \\ -1+t \end{array}\right\} \le m \le \left\{\begin{array}{l} 1-t \\ 1+t \end{array}\right. \tag{6.1}$$

$$\left.\begin{array}{l} -1+n \\ -1-n \\ -1+n-t+m \\ -1-n+t+m \end{array}\right\} \le m_\theta \le \left\{\begin{array}{l} 1+n \\ 1-n \\ 1+n-t+m \\ 1-n+t+m \end{array}\right. \tag{6.2}$$

In (6.2), the value of m_θ is of no concern and it is necessary only to be sure that each left-hand side is less than each right-hand side. Eliminating the redundant and tautological members of the resulting set of 16 inequalities plus the 4 inequalities of (4.1) we obtain the 12 inequalities

$$\begin{array}{c} \left.\begin{array}{l} -1 \\ -1+t \end{array}\right\} \le n \le \left\{\begin{array}{l} 1 \\ 1+t \end{array}\right. \\ -1+|t| \le m \le 1-|t| \\ -2+t \le 2n \pm m \le \pm 2+t \end{array} \tag{6.3}$$

Inequalities (6.3) define a polyhedron in an n, m, t space which is symmetric with respect to the m axis and the plane m = 0, and which inscribes the actual yield surface when the tensile yield stress is known. The specific polygons for positive constant t and positive m are

$$\underline{0 \le t \le \tfrac{1}{2}} \qquad\qquad \underline{\tfrac{1}{2} \le t \le 1}$$

$$n = -1 + t \qquad\qquad n = -1 + t$$

$$m = \begin{cases} 2-t+2n \\ 1-t \\ 2+t-2n \end{cases} \qquad\qquad m = 1 - t \tag{6.4}$$

$$n = 1 \qquad\qquad n = 1$$

The heavy curves in Fig. 6 show typical polygons (6.4).

A similar analysis may be made for the maximum reduced stress polyhedron. Elimination of m_θ from (5.16) leads to

$$\begin{gathered} -8+6|t| \le 3m \le 8-6|t| \\[1ex] \left.\begin{matrix} -2+t \\ -4+4t \\ -4-2t \end{matrix}\right\} \le 2n \le \left\{\begin{matrix} 2+t \\ 4+4t \\ 4-2t \end{matrix}\right. \\[1ex] \left.\begin{matrix} -12+10t \\ -12-2t \end{matrix}\right\} \le 8n \pm 3m \le \left\{\begin{matrix} 12+10t \\ 12-2t \end{matrix}\right. \\[1ex] -8+2t \le 4n \pm 3m \le 8+2t \end{gathered} \tag{6.5}$$

For an upper bound on the tension-test measured shell, we supplement (6.5) with the inequalities obtained from (5.17) to obtain the polygons

$$\underline{0 \le t \le 2/3} \qquad\qquad \underline{2/3 \le t \ge 1}$$

$$2n = -2 + t \qquad\qquad 2n = -4 + 4t$$

$$3m = \begin{cases} 12 - 10t + 8n \\ 4 \\ 12 - 2t + 8n \end{cases} \qquad\qquad 3m = \begin{cases} 12 - 10t + 8n \\ 8 - 6t \\ 12 - 2t - 8n \end{cases} \tag{6.6}$$

$$2n = 2 + t \qquad\qquad 2n = 4 - 2t$$

$$\underline{1 \le t \le 4/3}$$

$$2n = -4 + 4t$$

$$3m = 8 - 6t$$

$$2n = 4 - 2t$$

shown by the light curves in Fig. 6.

The bounding yield curves for a cylindrical shell have been applied to the problem of a cantilever shell under internal pressure [24]. Figure 7 shows the resulting bounds on the yield-point pressure as a function of the dimensionless parameter

$$\omega = L/\sqrt{AH}$$

where A is the radius and 2H the shell thickness. Similar results when the shearing yield stress is known are also given in [24].

7. LIMITATIONS AND EXTENSIONS

The aim of the preceding sections has been to give the theoretical background of a practical theory of plastic shells. Thus, we have attempted neither to give a complete catalogue of available problem solutions, nor to give the most general possible theory. With regard to problem solutions, a representative selection and extensive bibliography may be found in [10] .

Among the physically present physical concepts which have ~~not been considered in our idealized model are the interaction of~~ elastic and plastic strains, the effect of strain-hardening, changes in stress distributions due to small geometry changes induced by the loads, and nonlinearities due to finite strains. Hodge and his associates [25, 26, 27, 28] have considered various simple problems in which elastic strains and strain-hardening were included. In every case investigated, a representative load-deformation curve had the qualitative form shown in Fig. 8 [25] . Based upon these examples, it appears reasonable to assume that structures made of real material exhibit qualitatively different behavior depending upon whether the load is above or below the yield-point load of the same structure made of an idealized rigid/perfectly-plastic material. Thus, if the purpose of the investigation is to determine only the load value at which this qualitative difference occurs, the rigid/perfectly-plastic model considered herein will provide a reasonable estimate for the desired information. Further, Figure 8 indicates that if more detailed information is desired for loads less than the yield-point load, it is reasonable to neglect strain-hardening and consider an elastic/perfectly-plastic material. Finally, for information at loads above the yield-point load, elastic strains are relatively unimportant and one may use a rigid/strain-hardening material as a model.

The situation with regard to small and large geometry changes is less clear. It has been shown by Haythornthwaite [29] that the behavior of a beam whose ends are fully fixed as to both slope and separation is quite different from one whose ends are clamped but free to move towards each other under load. For the former, even a small deformation of the order of half the beam height introduces axial forces which substantially raise the load-carrying capacity of the beam. Similar results for circular plates were found by Onat and Haythornthwaite [30].

The beam and circular plate problem have proved solvable because of the fact that they deformed into easily characterized simple shapes. For other shell problems, the initial velocity field at the yield-point load predicts that elementary shell shapes such as spheres, cones, or cylinders, deform into complex shapes which are not easily characterized. Therefore, a general theory of the post-yield behavior of shells must probably await a more general technique for determining the yield-point load. In view of the complexity of solutions for such simple shapes as spheres or cones, it appears almost certain that any general approach must be primarily numerical. A first step in this direction has been taken by Onat and Lance [16], for a shallow conical shell, but it is not yet clear if their methods can be generalized.

With regard to truly large deformations, the small-strain theory presented herein is wholly inadequate. However, for very thin shells, it does appear reasonable in this case to neglect bending stresses entirely and construct a membrane theory of finite shell deformation. This has been done by Salmon [31] for an initially cylindrical shell. Obviously more work remains to be done in this area also.

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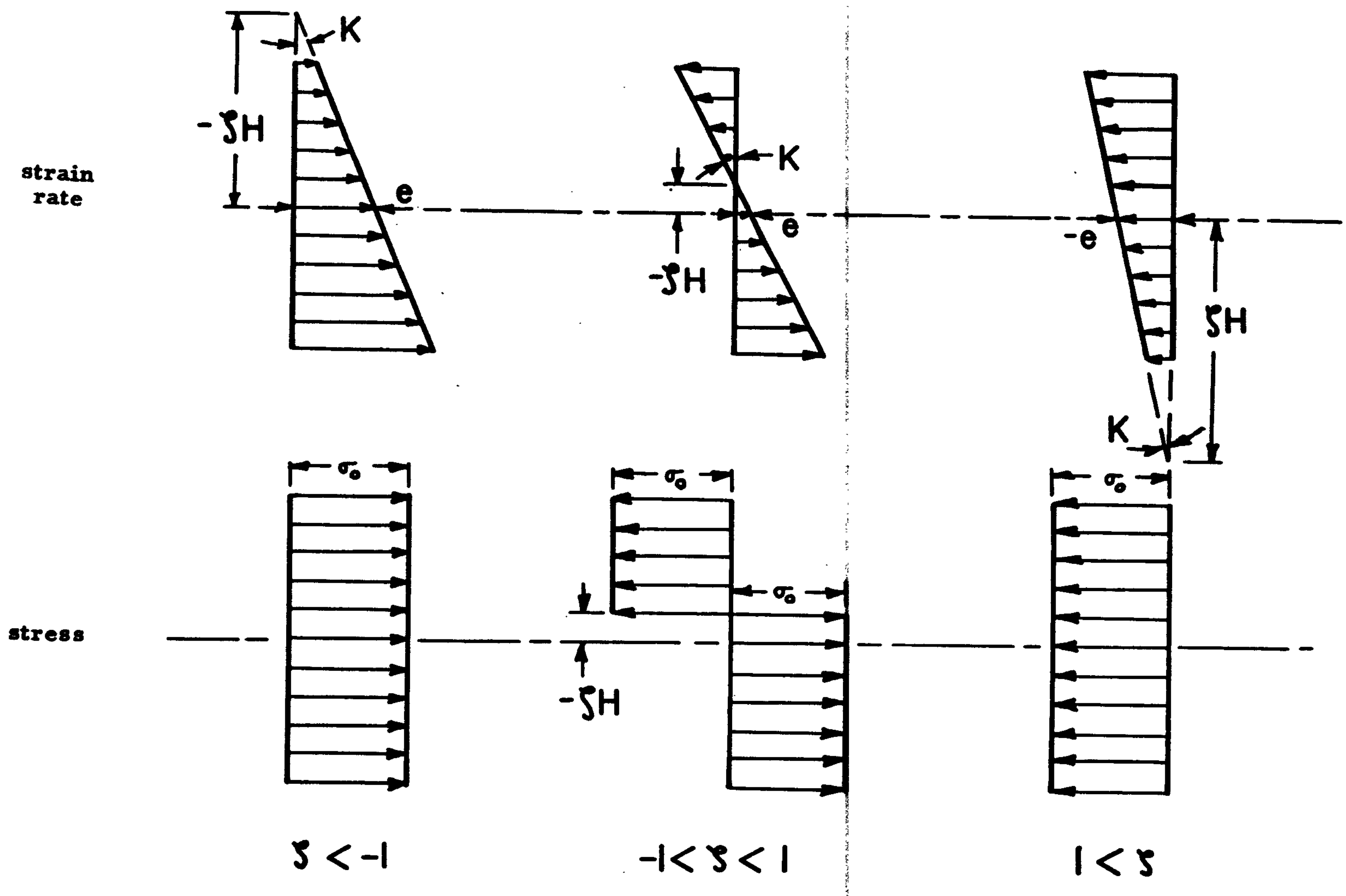


Figure 1. Strain rate and stress distributions for positive K

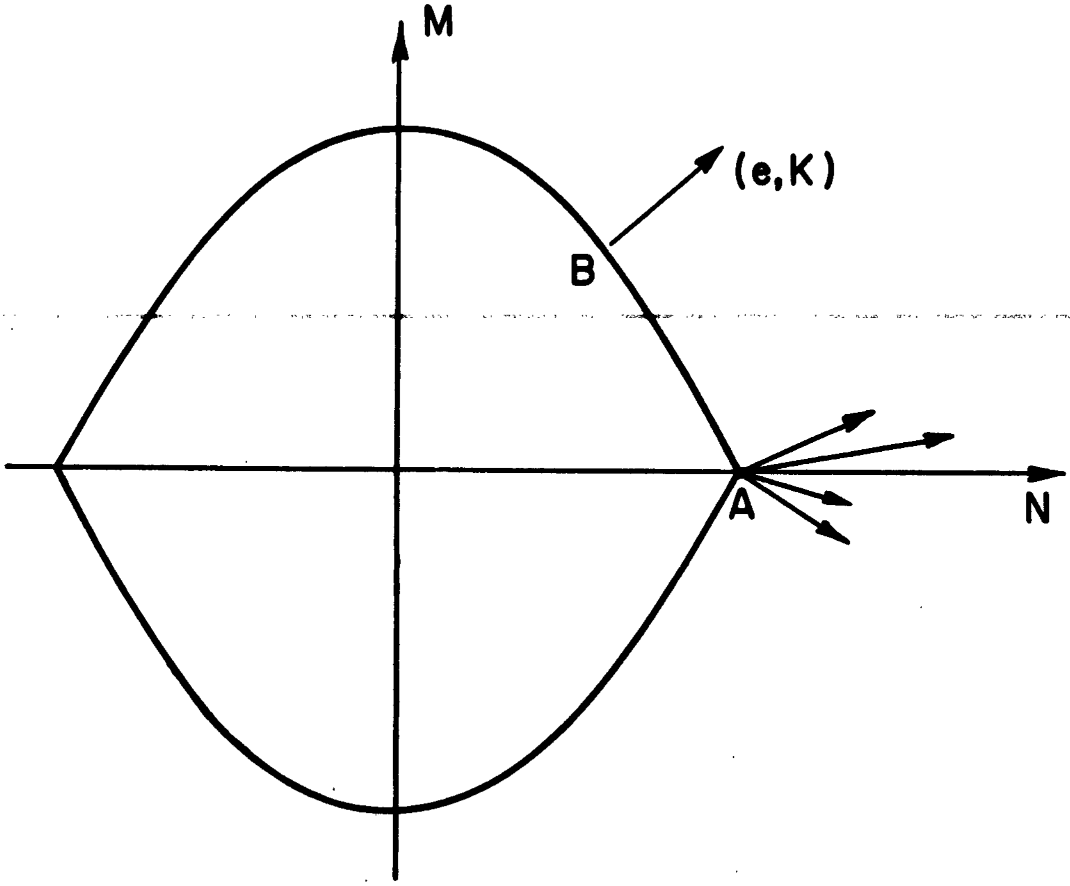


Figure 2. Yield curve for rectangular beam

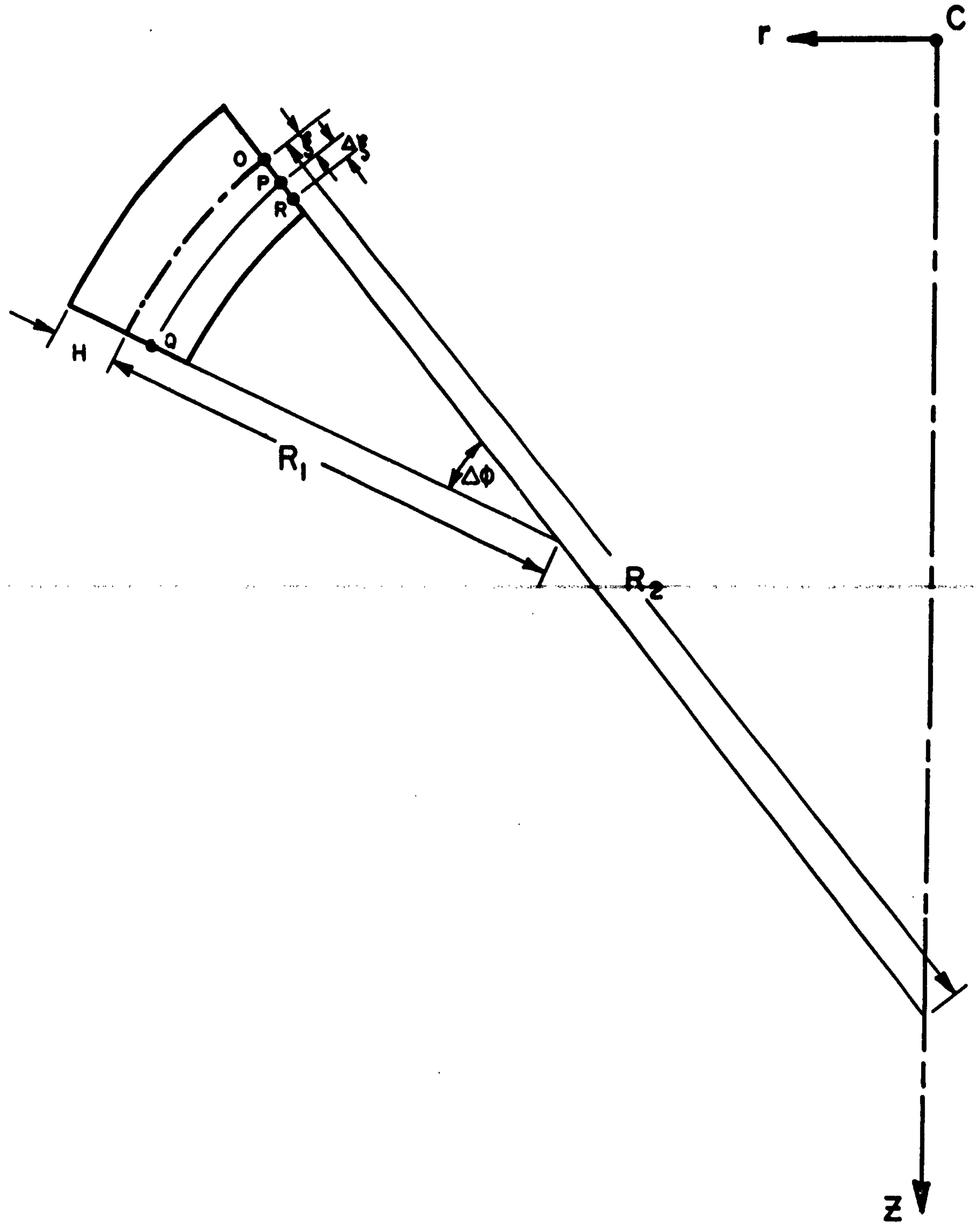


Figure 3. Shell element

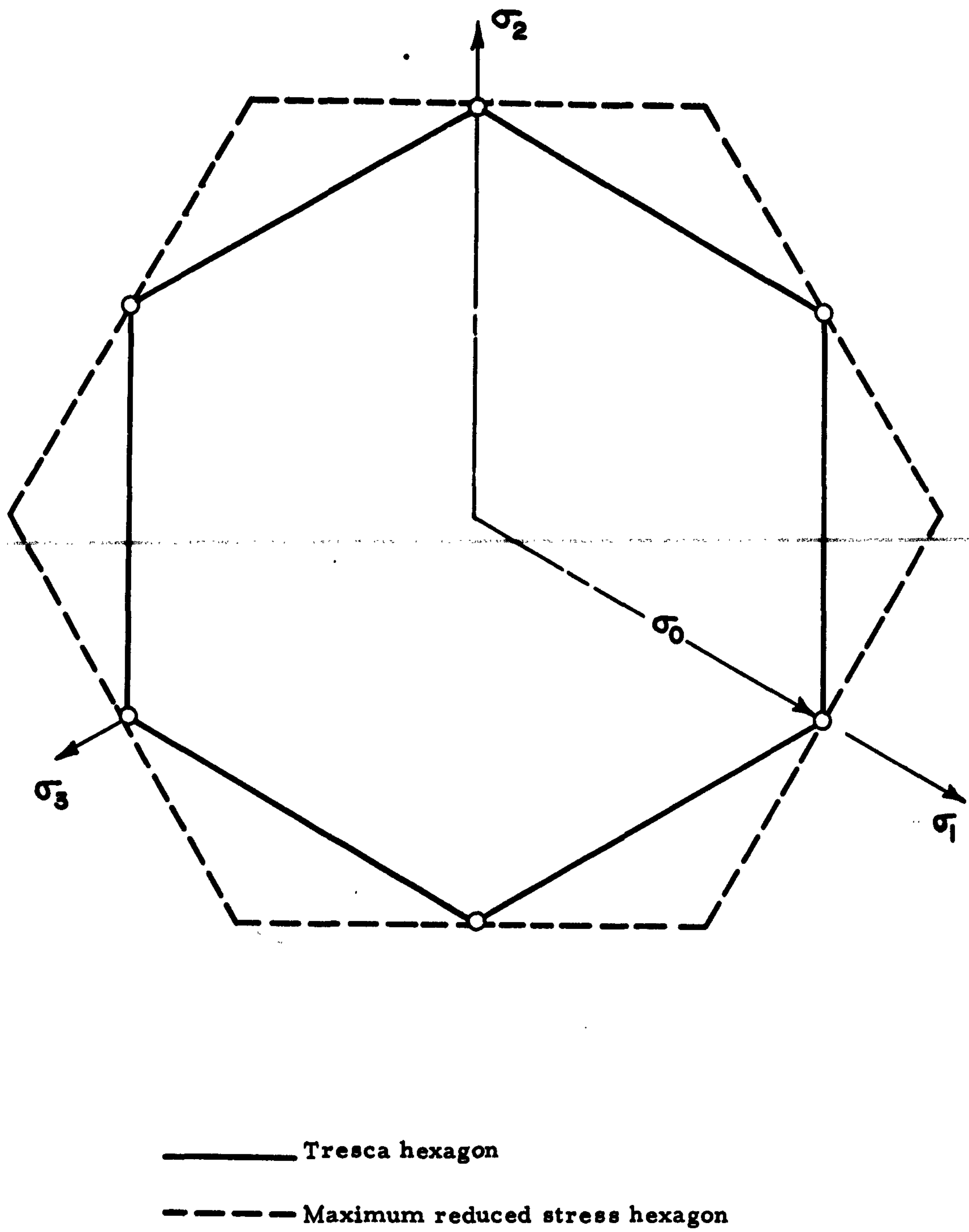


Figure 5. Piecewise linear yield conditions

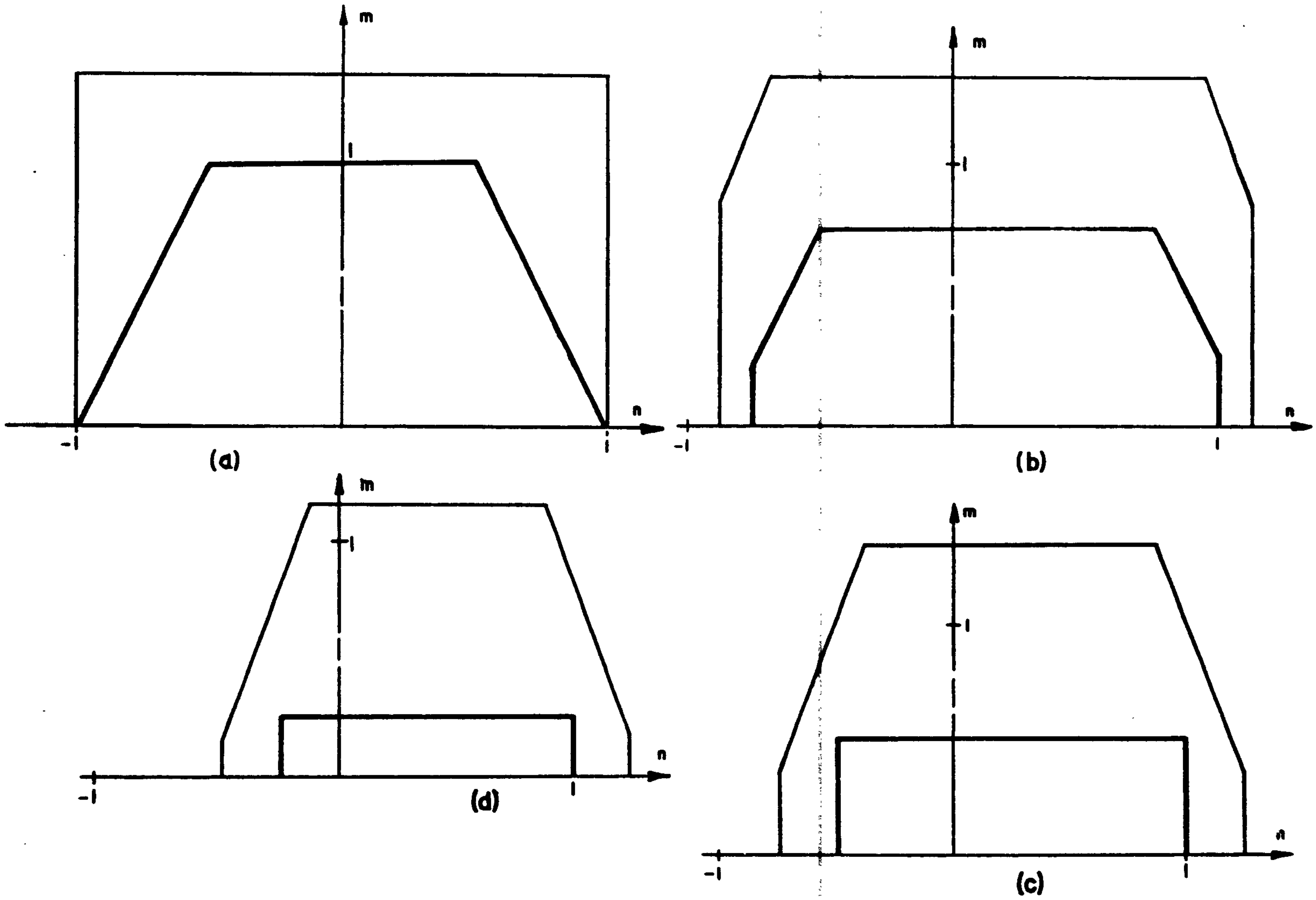


Figure 6. Yield polygons for cylindrical shell

(a) t = 0

(b) t = 1/4

(c) t = 1/2

upper bound, tensile stress known

lower bound, tensile stress known

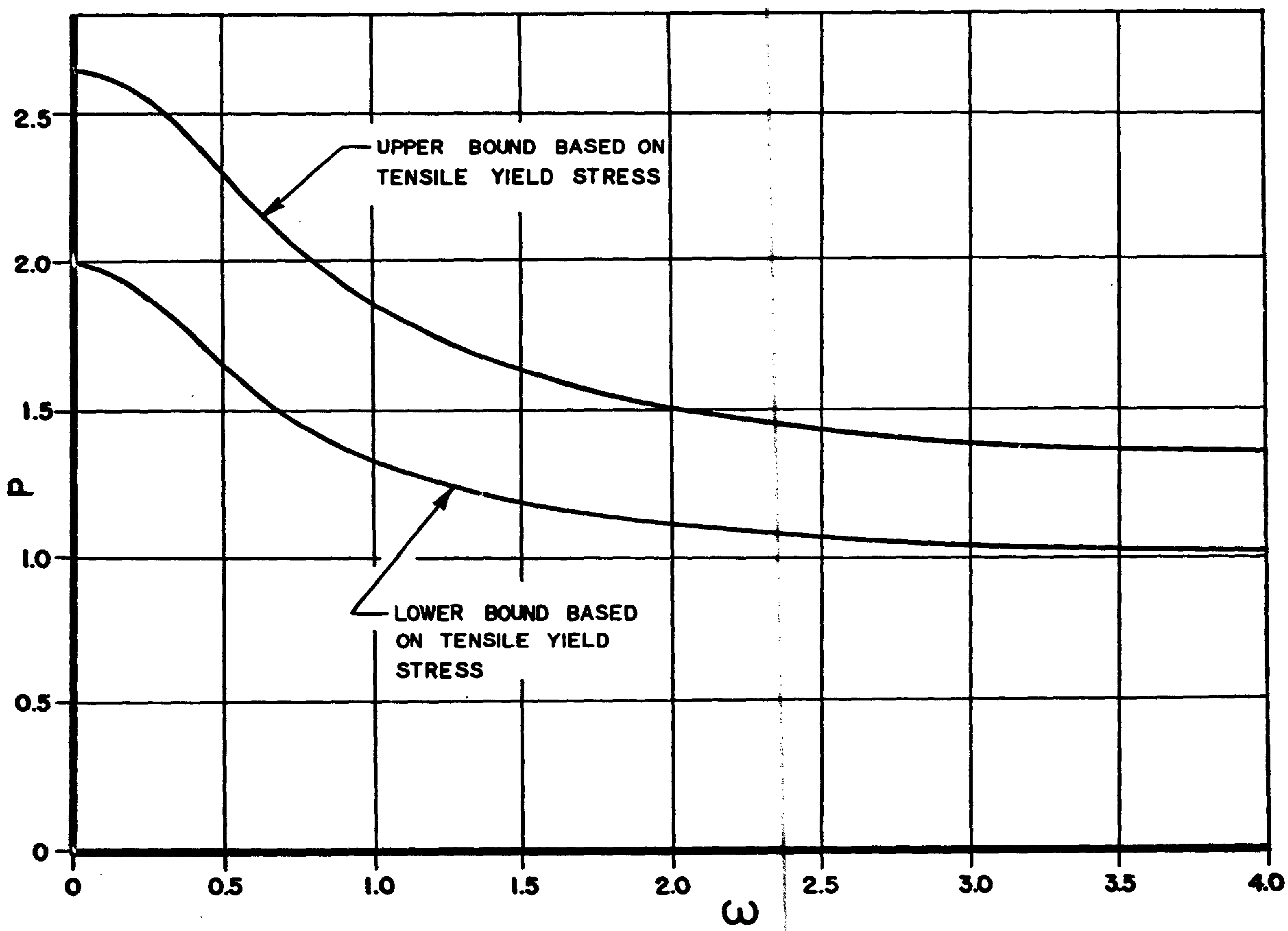
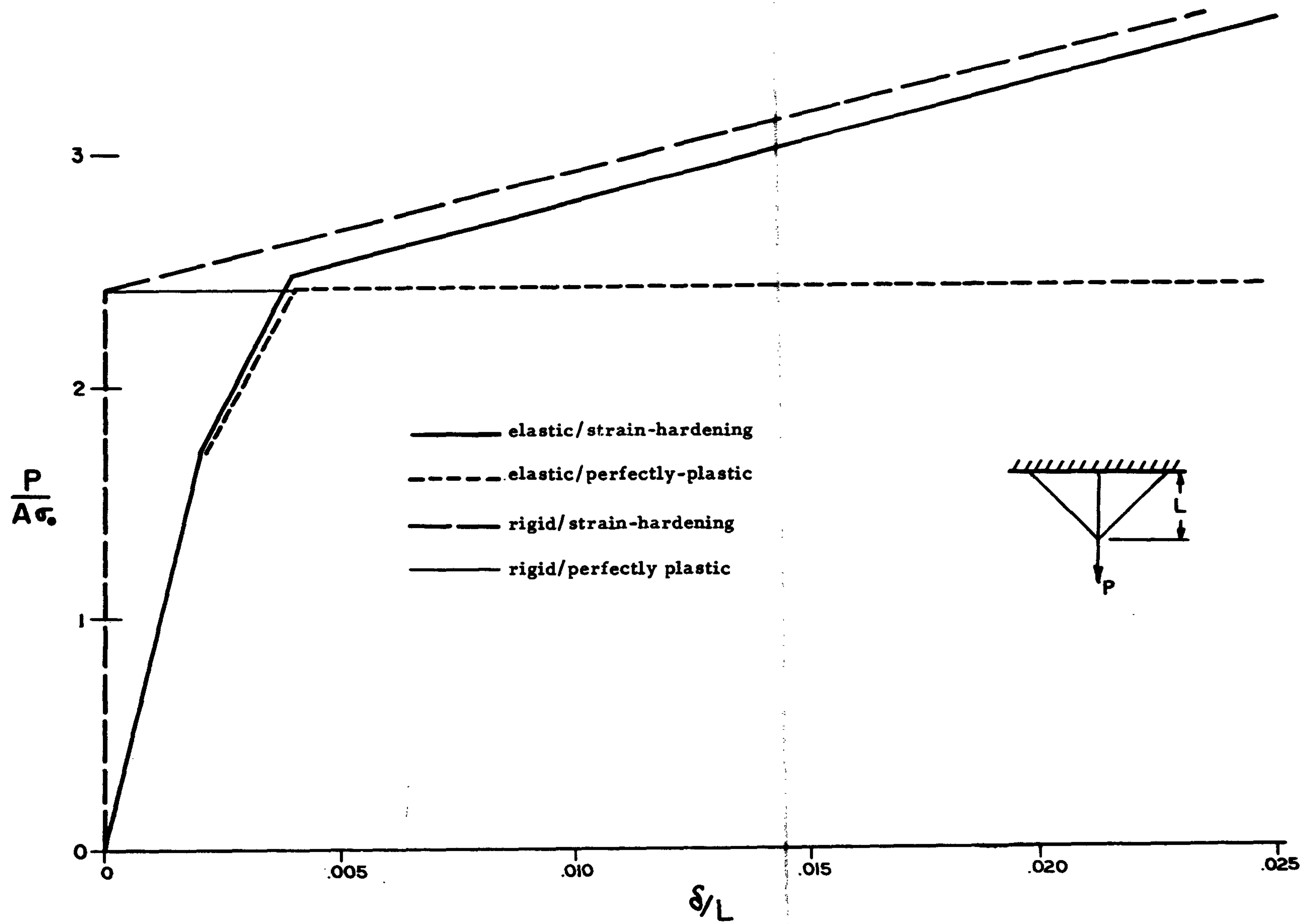


Figure 7. Bounds on yield-point load for cylindrical shell



3
2
1
0
$\frac{P}{A\sigma_0}$
0
.005
.010
.015
.020
.025
δ/L
elastic/strain-hardening
elastic/perfectly-plastic
rigid/strain-hardening
rigid/perfectly plastic
L
P

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